

Modeling sonoluminescence

Alan Chodos¹ and Sarah Groff²

¹*Center for Theoretical Physics, Yale University, 217 Prospect Street, New Haven, Connecticut 06511-8167*

²*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

(Received 4 August 1998; revised manuscript received 5 November 1998)

In single-bubble sonoluminescence, a bubble trapped by a sound wave in a flask of liquid is forced to expand and contract; exactly once per cycle, the bubble emits a very sharp (<50 ps) pulse of visible light. This is a robust phenomenon observable to the naked eye, yet the mechanism whereby the light is produced is not well understood. One model that has been proposed is that the light is “vacuum radiation” generated by the coupling of the electromagnetic fields to the surface of the bubble. In this paper, we simulate vacuum radiation by solving Maxwell’s equations with an additional term that couples the field to the bubble’s motion. We show that, in the static case originally considered by Casimir [Proc. K. Ned. Akad. Nel. **51**, 783 (1948)], we reproduce Casimir’s result. In a simple purely time-dependent example, we find that an instability occurs and the pulse of radiation grows exponentially. In the more realistic case of spherically symmetric bubble motion, we again find exponential growth in the context of a small-radius approximation. [S1063-651X(99)08803-0]

PACS number(s): 78.60.Mq, 03.50.-z, 03.70.+k, 03.50.De

I. INTRODUCTION

Single-bubble sonoluminescence [1] is a mysterious phenomenon. A small bubble of gas, usually air, is trapped at the center of a flask of liquid, usually water, by the application of an intense acoustic field. The frequency of the field is typically 25 or 30 kHz, and, once per cycle, driven by the sound field, the bubble undergoes expansion and then rapid contraction. If the parameters are right [2] (here “parameters” refers to such things as the intensity of the sound field, the concentration of the gas, the chemical composition of the gas, or the temperature of the water) the bubble will emit a very narrow pulse of light during the contraction phase of the cycle. The sound wave, with a time scale of tens of microseconds, produces a contraction of the bubble measured in tens of nanoseconds, which in turn somehow generates a pulse of visible light whose duration has recently been measured to be tens of picoseconds [3]. Furthermore, even though the motion of the bubble is quite violent, if the parameters are right it can be remarkably stable, repeating itself over millions of cycles, with the flash of light appearing at the same point in the cycle each time [4].

Other aspects of the phenomenology of sonoluminescence are also worthy of note. For example, the spectrum of emitted light is only partially known, because the water absorbs all wavelengths shorter than about 180 nm [5]. The part that is observed looks like the tail of a rising distribution, and attempts to fit it to a thermal spectrum have led to the speculation that the emitting region is very hot, certainly in excess of 25 000 K, and perhaps even as high as a million degrees, at which point nuclear fusion might be expected to be significant [6].

Another peculiarity is the fact that, whereas air-filled bubbles work well as a vehicle for sonoluminescence, bubbles filled with either oxygen or nitrogen, or indeed with a suitable mixture of these two gases, do not [7]. The small noble-gas component of air is essential for significant sonoluminescence to take place. This agrees with the suggestion that, during the first second or so of the bubble’s oscillation, the oxygen and nitrogen are ionized and absorbed by

the water, leaving a rarefied bubble filled with noble gas [8]. Experiments done with bubbles filled with various noble gases confirm that they produce sonoluminescence efficiently.

There is much more to the phenomenology of sonoluminescence, as described in a number of recent reviews [9]. Sonoluminescence is a complex phenomenon, involving as it does the motion of the bubble, the dynamics of the gas inside the bubble, and the mechanism that produces the flash of light. Our main concern in this work will be the last of these. The literature contains two classes of models to explain the flash of light. One involves the gas inside the bubble in an essential way [10]. There is no doubt that the gas undergoes compression and heating during the contraction phase of the bubble’s motion, and this type of explanation relies on either thermal radiation, or else bremsstrahlung, to produce the light.

The second type of explanation, on which we shall focus, is that the observed light is due to “vacuum radiation” [11–15], which is a dynamical counterpart to the well-known Casimir effect. In this view, given a particular motion of the bubble,

$$r = R(t) \quad (1)$$

[here r is the radial coordinate that describes the bubble’s surface, and $R(t)$ is a prescribed function of time; we assume a spherical bubble centered at the origin for simplicity], the radiation would take place even if the bubble were completely evacuated. The role of the gas, and in particular the special role that seems to be played by the noble gases, is merely to modulate the motion of the bubble, i.e., to give rise to a specific $R(t)$. It is then the motion of the boundary that directly gives rise to the radiation.

To explore this idea, our approach will be to take as given all of the physics associated with the motion of the bubble and the dynamics of the gas, and to extract therefrom the single function $R(t)$ which represents the experimentally measured bubble motion. Our next task is to construct a model in which the electromagnetic field is coupled to the bubble surface at $r = R(t)$.

One approach would be to attempt to derive this coupling from an examination of the dielectric properties of water; in practice, this would mean simply endowing the water with a dielectric constant ϵ , and letting

$$\epsilon(\vec{x}, t) = \epsilon\theta(r - R(t)) + \theta(R(t) - r).$$

Considerable attention in the literature has been devoted to the case of $R(t) = \text{const}$, which, although it obviously neglects the dynamical mechanism that turns Casimir energy into real photons, is supposed to provide an order of magnitude estimate of the energy available, which can then be compared to the energy that is produced in sonoluminescence. The question of whether the Casimir energy is sufficient in this respect has become a rather controversial one [18]. Attempts to treat this problem dynamically have led to interesting results, but have not fully resolved the issue [12,13,14,19].

In this work, we shall choose a coupling that is not derivable (at least by us) from a direct consideration of the underlying physics. Rather, the interaction is chosen both for its simplicity, and because it naturally leads to a coupling localized on the boundary $r = R(t)$. In addition, as we shall show below, when one considers the case of two static parallel plates (Casimir's original problem) one recovers precisely the original Casimir energy, and is therefore encouraged to hope that the model may be a valid representation of the dynamical situation as well.

In Sec. II we shall introduce the model, and derive the boundary conditions on \vec{E} and \vec{B} that it implies. In Sec. III, we look at two instructive cases that are not directly related to sonoluminescence: the case of static, parallel plates mentioned above, and the case of a strictly time-dependent source, with no spatial dependence. In this latter case, we shall discover the existence of unstable modes that can lead to production of radiation at unexpectedly large rates.

In Sec. IV, we tackle the case of greatest interest, the collapsing bubble. Even classically, we are unable to solve the equations exactly, but we develop an approximation scheme that relies on the fact that the radius of the bubble is small, in the sense that $R(t) \ll cT$, where T is a time characteristic of the width of the sonoluminescent pulse. In this approximation we find the same sort of unstable modes that existed in the purely time-dependent case. Section V is devoted to conclusions, and we have also included an appendix in which further properties of the $F\vec{F}$ interaction term are discussed.

II. MODEL

We consider the following Lagrange density [16]:

$$\mathcal{L} = -\frac{1}{4}[F_{\mu\nu}F^{\mu\nu} + f(x)F_{\mu\nu}\vec{F}^{\mu\nu}]. \quad (2)$$

Here $F_{\mu\nu}$ has its usual meaning $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is the four-vector electromagnetic potential, and $\vec{F}^{\mu\nu} = (1/2)\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric symbol on four indices, and $\epsilon^{0123} = 1$.

As is well known, $F_{\mu\nu}\vec{F}^{\mu\nu}$ is a total divergence

$$F_{\mu\nu}\vec{F}^{\mu\nu} = \partial_\mu[2\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma], \quad (3)$$

so up to a surface term \mathcal{L} can be written

$$\mathcal{L} = -\frac{1}{4}[F_{\mu\nu}F^{\mu\nu} - 2(\partial_\mu f)e^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma]. \quad (4)$$

Hence by choosing

$$f(x) = f_0\theta[g(x)], \quad (5)$$

we obtain $\partial_\mu f = f_0\partial_\mu g \delta(g)$, so the second term in \mathcal{L} represents the coupling of the electromagnetic field to the surface given by $g(x) = 0$. (f_0 is a dimensionless constant.) For the case of the sonoluminescing bubble, we would choose $g(\vec{x}, t) = r - R(t)$.

The equations of motion that follow from \mathcal{L} are

$$\partial_\mu[F^{\mu\nu} + f(x)\vec{F}^{\mu\nu}] = 0, \quad (6)$$

or, since $\partial_\mu\vec{F}^{\mu\nu} = 0$ identically,

$$\partial_\mu F^{\mu\nu} + (\partial_\mu f)\vec{F}^{\mu\nu} = 0. \quad (7)$$

If we define \vec{E} and \vec{B} in the usual way, we obtain the modified Maxwell equations

$$\vec{\nabla} \cdot \vec{E} + \vec{\nabla} f \cdot \vec{B} = 0, \quad (8)$$

$$\vec{\nabla} \times \vec{B} - \dot{\vec{E}} - f\dot{\vec{B}} - \vec{\nabla} f \times \vec{E} = 0 \quad (9)$$

together with $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} + \dot{\vec{B}} = 0$. With choice (5), we have $\dot{f} = f_0 n^0 \delta(g)$ and $\vec{\nabla} f = f_0 \vec{n} \delta(g)$, where $n_\mu = (\dot{g}, \vec{\nabla} g)$ is the 4-normal to the surface. To see what these equations entail, we write

$$\vec{E} = \vec{E}_1 \theta(g) + \vec{E}_2 \theta(-g), \quad (10a)$$

$$\vec{B} = \vec{B}_1 \theta(g) + \vec{B}_2 \theta(-g), \quad (10b)$$

and substituting into Eqs. (8) and (9), we find that the pair (\vec{E}_1, \vec{B}_1) satisfy the free Maxwell equations for $g > 0$, and likewise (\vec{E}_2, \vec{B}_2) satisfy them for $g < 0$. At $g = 0$, we have the boundary conditions

$$\vec{n} \cdot (\vec{E}_1 - \vec{E}_2) + f_0 \vec{n} \cdot \vec{B} = 0, \quad (11)$$

$$\vec{n} \times (\vec{B}_1 - \vec{B}_2) - n_0 (\vec{E}_1 - \vec{E}_2) - f_0 (n_0 \vec{B} + \vec{n} \times \vec{E}) = 0, \quad (12)$$

$$\vec{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0, \quad (13)$$

$$\vec{n} \times (\vec{E}_1 - \vec{E}_2) + n_0 (\vec{B}_1 - \vec{B}_2) = 0. \quad (14)$$

Notice that the second pair of equations (13) and (14) removes the ambiguity as to which values of \vec{B} and \vec{E} to use in the terms proportional to f_0 in the first pair of equations (11) and (12).

III. SPECIAL CASES

A. Parallel plates

Before dealing with the time-dependent case, let us explore the physical significance of our model by revisiting the case originally considered by Casimir [17], i.e., two infinite

parallel planes separated by a distance a . We take $g(\vec{x}) = z(z-a)$. The planes divide space into three regions, $z < 0$, $0 < z < a$ and $z > a$, and in each region we can choose a plane-wave solution to Maxwell's equations,

$$\vec{E} = e^{-i\omega t} [\vec{e} e^{i\vec{k}\cdot\vec{x}} + \vec{e}' e^{i\vec{k}'\cdot\vec{x}}], \quad (15)$$

where $\vec{k} = (k_1, k_2, k_3)$ and $\vec{k}' = (k_1, k_2, -k_3)$ and

$$\vec{B} = e^{-i\omega t} [\vec{b} e^{i\vec{k}\cdot\vec{x}} + \vec{b}' e^{i\vec{k}'\cdot\vec{x}}]. \quad (16)$$

We need both \vec{k} and \vec{k}' because the boundary conditions will mix them.

Maxwell's equations imply that

$$\vec{k} \times \vec{b} = -\omega \vec{e}, \quad \vec{k} \times \vec{e} = \omega \vec{b} \quad (17)$$

and

$$\vec{k}' \times \vec{b}' = -\omega \vec{e}', \quad \vec{k}' \times \vec{e}' = \omega \vec{b}', \quad (18)$$

in each of the three regions, which in turn imply that $\omega^2 = \vec{k}^2 = \vec{k}'^2$.

It is now a matter of implementing the boundary conditions at $z=0$ and at $z=a$. After some algebra, it is not hard to show that the content of these conditions reduces to

$$z=0: \quad b_3 + b'_3 = 0, \quad (19)$$

$$z=a: \quad b_3 e^{ik_3 a} + b'_3 e^{-ik_3 a} = 0 \quad (20)$$

(here b_3 means the third component of \vec{b} in the region $0 < z < a$, and similarly for b'_3) from which it follows that

$$k_3 a = n\pi, \quad n=0, \pm 1, \pm 2, \dots \quad (21)$$

This is exactly the same spectrum used by Casimir in his original paper, and therefore the Casimir energy δE will be the same as his result

$$\frac{\delta E}{L^2} = -\frac{\pi^2}{720a^3}, \quad (22)$$

where L^2 is the area of one of the plates.

B. Time dependent source

Armed with the knowledge that our model reproduces the static Casimir energy, we now proceed to another simple example, which is very different physically: we take $f(\vec{x}, t)$ to depend only on t :

$$\begin{aligned} f(x, t) &= 0, \quad t \leq 0 \\ &= gt, \quad 0 < t < T \\ &= gT, \quad t \geq T. \end{aligned} \quad (23)$$

Since f is not a step function, there is no bubble in this case. Rather, the source is turned on everywhere at once at $t=0$, and is turned off again at $t=T$ ($f = \text{const}$, is without physical consequence, because $F_{\mu\nu} \vec{F}^{\mu\nu}$ is a total divergence).

The equations of motion in this case are of course just the free Maxwell equations for $t < 0$ and $t > T$, whereas for $0 < t < T$, one of the Maxwell equations is modified:

$$\vec{\nabla} \times \vec{B} - \vec{E} = g\vec{B}. \quad (24)$$

One can now study a plane-wave solution, propagating, say, along the z axis. For $t < 0$ we write

$$\vec{B} = (\alpha \hat{x} + b \hat{y}) e^{i(kz - \omega t)}, \quad (25)$$

$$\vec{E} = (b \hat{x} - a \hat{y}) e^{i(kz - \omega t)},$$

with $k^2 = \omega^2$. At $t=0$, this will be matched to a solution of the form

$$\vec{B} = e^{ikz} [e^{-i\Omega t} (\alpha \hat{x} + \beta \hat{y}) + e^{i\Omega t} (\gamma \hat{x} + \delta \hat{y})], \quad (26)$$

$$\vec{E} = e^{ikz} [e^{-i\Omega t} (\beta \hat{x} - \alpha \hat{y}) + e^{i\Omega t} (\delta \hat{x} - \gamma \hat{y})],$$

where, because of the modification to Maxwell's equations, one has $\Omega^2 = \omega^2 = [k(k \pm g)]$.

Corresponding to each of these solutions is a particular polarization $\hat{C}_{\pm} = 1/\sqrt{2}[\hat{x} \mp i\hat{y}]$. When matched to the $t < 0$ solution, the expression for \vec{B} , $0 < t < T$, becomes

$$\begin{aligned} e^{-ikz} \vec{B} &= \left(\frac{k + \Omega_+}{2\Omega_+} \right) \frac{\alpha + ib}{\sqrt{2}} \hat{C}_+ e^{-i\Omega_+ t} \\ &\quad - \left(\frac{k - \Omega_+}{2\Omega_+} \right) \frac{a + ib}{\sqrt{2}} \hat{C}_+ e^{i\Omega_+ t} \\ &\quad + \left(\frac{k + \Omega_-}{2\Omega_-} \right) \frac{a - ib}{\sqrt{2}} \hat{C}_- e^{-i\Omega_- t} \\ &\quad - \left(\frac{k - \Omega_-}{2\Omega_-} \right) \frac{a - ib}{\sqrt{2}} \hat{C}_- e^{i\Omega_- t}, \end{aligned} \quad (27)$$

with a similar expression for \vec{E} .

One can extend this analysis by matching this solution to a suitable expression for \vec{E} and \vec{B} in the region $t > T$, where of course $\omega^2 = k^2$ again. But we shall not need this extension in what follows.

The feature most worthy of note is that (for $g > 0$) the frequency Ω_- becomes imaginary when $k < g$. (If $g < 0$, then Ω_+ becomes imaginary.) Hence \vec{E} and \vec{B} grow exponentially with time over the interval $0 < t < T$. We shall see below that, at least in a certain approximation, this feature persists in the case of a spherically oscillating bubble.

To quantize this model, we can express \vec{E} and \vec{B} in terms of a vector potential \vec{A} , and endow the Fourier coefficients of \vec{A} with the appropriate commutation relations. Effectively this means that the coefficients a and b in the above expressions become quantum operators. We must also generalize our solution to the case of a plane wave propagating in an arbitrary direction, but this is easily done since the z axis used above was in no way special. It is of interest to compute the rate of energy production per unit volume by the external source. We do this by forming the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}), \quad (28)$$

and taking the vacuum expectation value of its time derivative. \mathcal{H} is normal ordered, so that $\langle 0|\mathcal{H}|0\rangle = 0$ for $t < 0$. Since the expressions are quite lengthy, we simplify matters by retaining only those pieces which grow exponentially. After some calculation, we then find

$$\begin{aligned} \frac{d}{dt} \langle 0|\mathcal{H}|0\rangle_{\text{exp}} &= \theta(t) \theta(T-t) \frac{g^5}{16\pi^2} \\ &\times \int_0^1 \frac{x^{5/2}}{(1-x)^{1/2}} e^{2gt\sqrt{x(1-x)}} dx, \end{aligned} \quad (29)$$

where the notation exp on the matrix element means the exponentially growing piece. A simple stationary-phase estimate of the integral gives

$$\frac{d}{dt} \langle 0|\mathcal{H}|0\rangle_{\text{exp}} \approx \theta(t) \theta(T-t) \frac{g^5}{64\pi^2} e^{gt}. \quad (30)$$

We can try to connect this to sonoluminescence (despite the fact that there is no bubble) by choosing $T = 10^{-11}$ sec (the duration of a typical pulse) and $1/g = 2 \times 10^{-7}$ m (the cutoff on the observed spectrum). We then find $gT = 1.6 \times 10^4$, which, needless to say, produces a huge number when inserted into the exponent in Eq. (30).

At this point, we can simply argue that our model is too far removed from the phenomenology of sonoluminescence to be expected to give reasonable results. Later, however, we shall have to deal with this question in the context of a more realistic model, to which we now turn.

IV. COLLAPSING BUBBLE

We take $f(x) = f_0 \theta(r - R(t))$. Our strategy will be to attempt to solve the classical problem, looking for the kind of exponential behavior in time that we found in the previous example. If this is indeed found, then, reasoning by analogy with the previous example, we will argue that, when quantized, the model will produce an exponentially growing pulse of vacuum radiation over some period of time.

Because of the spherical symmetry, it is appropriate to expand \vec{E} and \vec{B} in terms of vector spherical harmonics. Different values of l and m will not couple to each other. We write

$$\vec{E} = e_1 \vec{L} Y_{lm} + (re'_2 + e_2) \vec{\nabla} Y_{lm} + \frac{l(l+1)}{r^2} e_2 \vec{r} Y_{lm} \quad (31)$$

and

$$\vec{B} = b_1 \vec{L} Y_{lm} + (rb'_2 + b_2) \vec{\nabla} Y_{lm} + \frac{l(l+1)}{r^2} b_2 \vec{r} Y_{lm}. \quad (32)$$

Here $\vec{L} = (1/i) \vec{r} \times \vec{\nabla}$, and the e 's and b 's are functions of r and t ; we cannot separate variables any further because the boundary conditions will mix r and t .

These forms automatically satisfy $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0$. The rest of Maxwell's equations imply that

$$\ddot{\varphi} - \varphi'' - \frac{2}{r} \varphi' + \frac{l(l+1)}{r^2} \varphi = 0, \quad (33)$$

where φ is any of the e 's or b 's, and furthermore that

$$e_1 = i\dot{b}_2 \quad \text{and} \quad b_1 = -i\dot{e}_2. \quad (34)$$

In these equations, a prime is $\partial/\partial r$, and an overdot is $\partial/\partial t$. So if we cast the boundary conditions entirely in terms of b_2 and e_2 , we can recover e_1 and b_1 from Eq. (34).

In fact it is not hard to express the boundary conditions in terms of e_2 and b_2 . We expand \vec{E} and \vec{B} as in Eqs. (31) and (32) separately inside and outside the bubble, and we let $\Delta\varphi = \varphi_{\text{out}} - \varphi_{\text{in}}$, where once again φ is any of the e 's or b 's. Then we find, at $r = R(t)$,

$$\Delta b_3 = 0, \quad (35)$$

$$\Delta e_2 = -f_0 b_2, \quad (36)$$

$$\Delta \dot{b}_2 = \frac{-f_0 \dot{R}}{1 - \dot{R}^2} \left[\frac{e_2}{R} + e'_2 + \dot{R} \dot{e}_2 \right], \quad (37)$$

and

$$\Delta \dot{e}_2 = \frac{-f_0}{1 - \dot{R}^2} \left[\frac{R b_2}{R} + \dot{b}_2 + \dot{R} b'_2 \right]. \quad (38)$$

One can show that the expressions on the right-hand side of these equations all have zero discontinuity at $r = R(t)$, so there is no ambiguity as to which values to insert.

For simplicity, we choose to analyze the case $l = 1$. Then the most general solutions to Eq. (33), for b_2 and e_2 , are

$$\begin{aligned} b_2^{\text{out}} &= \frac{\partial}{\partial r} \left[\frac{1}{r} (\tilde{\beta}(t+r) - \hat{\beta}(t-r)) \right], \\ b_2^{\text{in}} &= \frac{\partial}{\partial r} \left[\frac{1}{r} (\beta(t+r) - \beta(t-r)) \right], \\ e_2^{\text{out}} &= \frac{\partial}{\partial r} \left[\frac{1}{r} (\tilde{\gamma}(t+r) - \hat{\gamma}(t-r)) \right], \\ e_2^{\text{in}} &= \frac{\partial}{\partial r} \left[\frac{1}{r} (\gamma(t+r) - \gamma(t-r)) \right]. \end{aligned} \quad (39)$$

Here the β 's and γ 's are arbitrary functions of the indicated arguments. In writing these equations, we have imposed the requirement that b_2 and e_2 be regular at $r = 0$. The functions $\tilde{\beta}$ and $\tilde{\gamma}$ determine the waves propagating inward from infinity, and should be taken as initial data. In principle, it should be possible to use the boundary conditions, [Eqs. (35)–(38)] to determine the inside solution, specified by β and γ , and the outgoing waves specified by $\hat{\beta}$ and $\hat{\gamma}$. The effect we are looking for is to see whether nonexponentially growing incoming data $\tilde{\beta}$ and $\tilde{\gamma}$ can generate exponential growth in the outgoing solution $\hat{\beta}$ and $\hat{\gamma}$.

Unfortunately, when we substitute forms (39) into the boundary conditions, we find rather complicated functional

difference equations that we do not know how to solve. Instead we rely on an approximation that is based on the following observation. Phenomenologically, the smallest time scale we are interested in is the width of the sonoluminescent pulse, $\Delta t_{\min} \cong 10^{-11}$ sec. The largest length scale we are interested in is the maximum size of the bubble, $R_{\max} \cong 10^{-4}$ m. But, in units with $c=1$,

$$\frac{R_{\max}}{\Delta t_{\min}} = 10^7 \text{ m/sec} = \frac{1}{30} \quad (40)$$

(this is actually a generous overestimate, since sonoluminescence occurs when the bubble radius is at least an order of magnitude smaller than R_{\max}). Thus it might make sense to regard $R(t)$ as a small parameter, and to expand our equations accordingly. We do this as follows: in the expressions for b_2 and e_2 [Eq. (39)] we replace the arguments $t \pm r$ by $t \pm \epsilon r$, where ϵ is a bookkeeping parameter in which we perform a systematic expansion. At the end, we set $\epsilon=1$.

To obtain a consistent expansion, we not only expand the arguments of the functions, we must also expand the functions themselves:

$$\varphi(t) = \varphi_0(t) + \epsilon \varphi_1(t) + \epsilon^2 \varphi_2(t) + \dots, \quad (41)$$

where φ stands for any of the unknowns $\hat{\beta}$, $\hat{\gamma}$, β , and γ . The input functions $\tilde{\beta}$ and $\tilde{\gamma}$ are regarded as known, and so are not expanded. The advantage of this expansion procedure is that we thereby obtain relations among functions all of which are evaluated at the same argument t . Henceforth we denote d/dt by a prime.

To obtain nontrivial results, we must retain terms up to order ϵ^3 . For convenience, we introduce the notation $\rho_1(t) = \beta_0'''(t)$ and $\rho_2(t) = \gamma_0'''(t)$. We find

$$\begin{aligned} \hat{\beta}_0 &= \tilde{\beta}, & \hat{\gamma}_0 &= \tilde{\gamma}, \\ \hat{\beta}_1 &= \hat{\beta}_2 = \hat{\gamma}_1 = \hat{\gamma}_2 = 0, \\ \hat{\beta}_3 &= \frac{2}{3} R^3 [\rho_1 - \tilde{\beta}'''], \\ \hat{\gamma}_3 &= \frac{2}{3} R^3 [\rho_2 - \tilde{\gamma}''' - f_0 \rho_1] \end{aligned} \quad (42)$$

and

$$\begin{aligned} \rho_1 &= \tilde{\beta}''' - \frac{1}{3} \left(\frac{f_0}{1 - \dot{R}^2} \right) [2\rho_2 + R\dot{R}\rho_2'], \\ \rho_2 &= \tilde{\gamma}''' - \frac{1}{3} \left(\frac{f_0}{1 - \dot{R}^2} \right) [(1 - 3\dot{R}^2)\rho_1 + R\dot{R}\rho_1']. \end{aligned} \quad (43)$$

The functions whose behavior we want to study are $\hat{\beta}_3$ and $\hat{\gamma}_3$, which give the first nontrivial corrections to the outgoing waves $\hat{\beta}$ and $\hat{\gamma}$. Our strategy will be to solve Eq. (43) for ρ_1 and ρ_2 , and then evaluate $\hat{\beta}_3$ and $\hat{\gamma}_3$ from Eq. (42). If we make the further approximation $\dot{R}^2 \ll 1$, which must surely be true for any realistic bubble motion, we can drop the \dot{R}^2 terms on the right hand side of Eq. (43), which can then be rewritten as

$$\frac{-f_0 R \dot{R}}{3} \begin{pmatrix} \rho_1' \\ \rho_2' \end{pmatrix} = M \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} \tilde{\gamma}''' \\ \tilde{\beta}''' \end{pmatrix}, \quad (44)$$

where M is the matrix $\begin{pmatrix} (1/3)f_0 & -1 \\ -1 & (-2/3)f_0 \end{pmatrix}$. The eigenvalues of M are

$$\lambda_{\pm} = \frac{1}{2} [\pm \sqrt{f_0^2 + 4} - \frac{1}{3} f_0]. \quad (45)$$

Note that $\lambda_+ > 0$, $\lambda_- < 0$. Let θ be the orthogonal matrix

$$\theta = \frac{1}{\sqrt{2}} \begin{bmatrix} k_+ & -k_- \\ k_- & k_+ \end{bmatrix}, \quad (46)$$

where $k_{\pm} = [1 \pm f_0 / \sqrt{f_0^2 + 4}]^{1/2}$. Then, setting $\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \theta \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$, our equation becomes

$$\frac{-f_0 R \dot{R}}{3} \begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix} = \begin{pmatrix} \lambda_+ \sigma_1 \\ \lambda_- \sigma_2 \end{pmatrix} + \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix}, \quad (47)$$

where $\begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \theta \begin{pmatrix} \tilde{\gamma}''' \\ \tilde{\beta}''' \end{pmatrix}$. Thus we have to solve

$$\frac{-f_0 R \dot{R}}{3} \sigma' = \lambda \sigma + \kappa, \quad (48)$$

whose solution is

$$\begin{aligned} \sigma(t) &= \exp \left[- \int_{t_0}^t dt' \frac{3\lambda}{f_0 R \dot{R}} \right] \\ &\times \left[- \int_{t_0}^t dt' \exp \left\{ \int_{t_0}^{t'} dt'' \frac{3\lambda}{f_0 R \dot{R}} \right\} \begin{pmatrix} 3\kappa \\ f_0 R \dot{R} \end{pmatrix} + \sigma(t_0) \right]. \end{aligned} \quad (49)$$

There are two such solutions, one for λ_+ and one for λ_- . Generally, the exponential behavior that is manifest on the right hand side of Eq. (49) will cancel in the first term, but will survive in the term proportional to $\sigma(t_0)$. Because in the region of interest we have $R\dot{R} < 0$, it will be λ_+ that gives the exponentially growing behavior (for $f_0 > 0$).

How do we interpret this result? First, we must recognize that the approximation we have made is potentially very dangerous, because the highest derivative in Eq. (48) is multiplied by a factor $-f_0 R \dot{R} / 3$, which we expect to be quite small, and indeed which we expect to go to zero for $|t|$ large. [Here we are restricting ourselves to only one cycle of the bubble's motion, so we take $R(t) \rightarrow \text{const}$ as $|t| \rightarrow \infty$.] As a consequence, it appears from the solution that the exponential behavior becomes more pronounced the smaller $R\dot{R}$ becomes, whereas we know from the original equation (48) that for $R\dot{R}$ strictly zero the solution is just $\sigma = -(1/\lambda)\kappa$, which exhibits no exponential behavior at all.

Our response to this is to imagine that for $|f_0 R \dot{R}|$ below some threshold value, it is indeed negligible, and therefore $\sigma = -(1/\lambda)\kappa$. At some time t_0 , $|f_0 R \dot{R}|$ crosses the threshold, and solution (49) kicks in. We therefore fix the arbitrary constant $\sigma(t_0)$ to be $-(1/\lambda)\kappa(t_0)$. As we have already ob-

served, for one of the two choices of λ , σ will be an exponentially growing function [except for the unlikely possibility that $\kappa(t_0)=0$].

The good news is that, within the context of our approximation, we have found the exponential behavior that we are looking for. The bad news is that, just as in the earlier, simpler example, we have an embarrassment of riches. There appears to be no mechanism within the model for turning the exponential behavior off—we have already fixed the one free parameter $\sigma(t_0)$. Let us assume that there is a dissipative mechanism, having to do with the properties of the gas inside the bubble or the liquid outside it, that we should add to our model. This will abort the vacuum radiation after some characteristic time. Because the exponential growth produces so much radiation, we must assume that the abortion takes place after only about a femtosecond. In this picture, therefore, the observed pulse really consists of an exceedingly rapid exponential rise of duration a femtosecond or so, followed by a relatively slow fall lasting tens of picoseconds, for which the dissipative mechanism is responsible.

V. CONCLUSIONS

It has been argued in the literature that vacuum radiation cannot be the source of sonoluminescence because the static Casimir energy is so small. Without entering into the controversy over how large the static Casimir energy is for a spherical bubble [18], we believe that, as illustrated by our model (which, after all, correctly reproduces the static Casimir energy in the case of parallel plates) there are two additional factors that ought to be taken into account: (1) The Casimir effect arises essentially from the coupling of the electromagnetic field to a boundary. When that boundary is moving, the field is coupled to a time-dependent source, which in and of itself leads to the production of energy. (2) If this time-dependent coupling gives rise to unstable modes, as it does in our model, then an unexpectedly large amount of energy can be produced.

The present work raises a number of issues for further investigation. Perhaps most important is tightening up the approximate treatment we have given for the classical solutions in the case of the collapsing bubble. It would be preferable to have a method of analysis that would conclusively demonstrate whether the period of exponential growth exists, and whether the model contains not only a mechanism for turning on the pulse but also for turning it off. Failing that, it will probably be necessary to include additional physics having to do with the kind of dissipation mechanism discussed above, capable of damping the vacuum radiation to a level compatible with what is seen experimentally. If this is the case, then the shape of the sonoluminescent pulse should exhibit a very rapid rise followed by a much slower decay.

Within the context of the expansion employed in this paper, as might be expected one finds that the higher orders in ϵ become progressively more complicated. We have examined the next nontrivial term, which is ϵ^5 , and we have verified that it does not qualitatively change the exponential behavior found in order ϵ^3 . We have not checked, however, that the ϵ^5 contribution is numerically small compared to the ϵ^3 contribution.

It would also be useful to quantize the electromagnetic field in the presence of the collapsing bubble, much as we

did for the simpler purely time-dependent case. We believe that any exponential behavior in the classical system will persist in its quantum counterpart, but having the explicit expression for quantum vacuum radiation would allow one to compare the details of the photon spectrum with experiment. It would also be useful, for numerical work, to have a good analytical approximation to $R(t)$.

Ultimately, electromagnetic radiation can be produced only by charges in motion. In the case of sonoluminescence, whether those charges effectively reside at the boundary of the bubble, as we contend in this paper, or within the gas inside the bubble, is a question that still awaits definitive resolution.

ACKNOWLEDGMENTS

We thank Matt Lippert and Andreas Nyffeler for their help in the early stages of this work. We are grateful to Bob Apfel and Jeff Ketterling for much enlightenment about the experimental aspects of sonoluminescence. This research was supported in part by U.S. DOE Grant No. DE-FG02-92ER-40704.

APPENDIX: MORE ABOUT $F\tilde{F}$

In this appendix we add a few remarks about the properties of $F\tilde{F}$. The $f(x)F\tilde{F}$ interaction analyzed in the text was chosen as the most convenient form for coupling the electromagnetic field to the bubble boundary. Whether it correctly captures the essential physics of this coupling is a matter that will require further investigation.

Another way of writing $F\tilde{F}$ is just $2\vec{E}\cdot\vec{B}$. It is, apart from the familiar $F_{\mu\nu}F^{\mu\nu}=1/2(\vec{B}^2-\vec{E}^2)$, the only Lorentz invariant that can be constructed from \vec{E} and \vec{B} by algebraic means. A term in the Lagrangian of the form $\theta F_{\mu\nu}\tilde{F}^{\mu\nu}$, with θ constant, has no physical consequence in an Abelian gauge theory such as electrodynamics, because, as noted in Eq. (3), it is a total divergence. In a non-Abelian theory (such as quantum chromodynamics or the electroweak theory) this term is still a divergence, but it nevertheless gives rise to nonperturbative physical effects because of the nontrivial topological structures, called instantons, that exist in such theories.

The anomalous divergence of the $U(1)$ axial vector current is proportional to $F\tilde{F}$. This term is directly responsible for the decay of the π^0 meson into two γ rays, which is its dominant decay mode.

In string theory, because of a property called S duality [20,21], two seemingly different theories can in fact be equivalent. This can be very useful because often one of the two theories is strongly coupled (and therefore intractable) whereas the other is weakly coupled. As it turns out, the system we have been studying in connection with sonoluminescence exhibits a simple form of S duality. To see this, it is useful to consider a slightly more general Lagrangian:

$$\mathcal{L} = -\frac{1}{4}[\varphi_1(x)F_{\mu\nu}F^{\mu\nu} + \varphi_2(x)F_{\mu\nu}\tilde{F}^{\mu\nu}]. \quad (\text{A1})$$

In the text we had $\varphi_1=1$ and $\varphi_2=f(x)$. In string theory,

$\varphi_1(x)$ would be related to the dilaton, whereas $\varphi_2(x)$ is known as the axion field.

This system is governed by two sets of equations.

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad (\text{A2})$$

and

$$\partial_\mu [\varphi_1 F^{\mu\nu} + \varphi_2 \tilde{F}^{\mu\nu}] = 0. \quad (\text{A3})$$

If we express $F_{\mu\nu}$ in the usual way as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (\text{A4})$$

then Eq. (A2) is an identity, while Eq. (A3) is a dynamical equation obtained by varying the Lagrangian with respect to A_μ .

We define

$$\tilde{G}^{\mu\nu} = \varphi_1 F^{\mu\nu} + \varphi_2 \tilde{F}^{\mu\nu}. \quad (\text{A5})$$

In Minkowski space, as is easily shown, the dual of a dual is the negative of the original tensor. Therefore,

$$G_{\mu\nu} = -\varphi_1 \tilde{F}_{\mu\nu} + \varphi_2 F_{\mu\nu}. \quad (\text{A6})$$

We can invert these relationships to obtain F and \tilde{F} in terms of G and \tilde{G} , and then substitute them into \mathcal{L} :

$$\mathcal{L} = -\frac{1}{4} \left(\frac{1}{\varphi_1^2 + \varphi_2^2} \right) \{ -\varphi_1 G_{\mu\nu} G^{\mu\nu} + \varphi_2 G_{\mu\nu} \tilde{G}^{\mu\nu} \}. \quad (\text{A7})$$

If we set

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad (\text{A8})$$

and vary \mathcal{L} with respect to B_μ , we obtain

$$\partial_\mu \left[\frac{-\varphi_1}{\varphi_1^2 + \varphi_2^2} G^{\mu\nu} + \frac{\varphi_2}{\varphi_1^2 + \varphi_2^2} \tilde{G}^{\mu\nu} \right] = 0, \quad (\text{A9})$$

and also

$$\partial_\mu \tilde{G}^{\mu\nu} = 0 \quad (\text{A10})$$

as an identity. It is straightforward algebra to show that Eq. (A9) is the same as Eq. (A2), and Eq. (A10) is, by definition, the same as Eq. (A3). Thus the physical content of the Lagrangian (A7) is the same as Eq. (A1), but the dynamical equation in one case is an identity in the other case, and vice versa. We see that if we can solve a system with sources (φ_1, φ_2) , then by duality we automatically obtain a solution with sources $[-\varphi_1/(\varphi_1^2 + \varphi_2^2), \varphi_2/(\varphi_1^2 + \varphi_2^2)]$. It is not clear, however, whether practical use can be made of this observation in the case of sonoluminescence.

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